

A NON-GOLOD RING WITH A TRIVIAL PRODUCT ON ITS KOSZUL HOMOLOGY

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ABSTRACT. We present a monomial ideal $\mathfrak{a} \subset S$ such that S/\mathfrak{a} is not Golod, even though the product on its Koszul homology is trivial. This constitutes a counterexample to a well-known theorem by Berglund and Jöllenbeck (the error can be traced to a mistake in an earlier article by Jöllenbeck). As far as we know, this is the first example of any ring (monomial or not) with a trivial products on its Koszul homology which is not Golod.

On the positive side, we show that if R is a monomial ring such that the r -ary Massey product vanish for all $r \leq \max(2, \operatorname{reg} R - 2)$, then R is Golod. In particular, if R is the Stanley-Reisner ring of a simplicial complex of dimension at most 3, then R is Golod if and only if the product on its Koszul homology is trivial.

Moreover, we show that if Δ is a triangulation of a \mathbb{k} -orientable manifold whose Stanley-Reisner ring is Golod, then Δ is 2-neighborly. This extends a recent result of Iriye and Kishimoto.

1. INTRODUCTION

Let $S = \mathbb{k}[x_1, \dots, x_n]$ be a polynomial ring over some field \mathbb{k} and let $\mathfrak{a} \subset S$ be a homogeneous ideal. The *Betti-Poincaré* series of $R := S/\mathfrak{a}$ is the formal power series

$$P_{\mathbb{k}}^R(t, z) := \sum_{j \geq 0} \sum_{i \geq 0} \dim_{\mathbb{k}} \operatorname{Tor}_j^R(\mathbb{k}, \mathbb{k})_i t^j z^i,$$

where $\operatorname{Tor}_j^R(\mathbb{k}, \mathbb{k})_i$ denotes the homogeneous component of $\operatorname{Tor}_j^R(\mathbb{k}, \mathbb{k})$ in degree i . The ring R is called *Golod* if the following holds:

$$(1) \quad P_{\mathbb{k}}^R(t, z) = \frac{(1 + tz)^n}{1 - \sum_{j \geq 1} \sum_{i \geq 0} \dim_{\mathbb{k}} H_j(K_R)_i t^{j+1} z^i}.$$

where K_R denotes the Koszul complex of R . In general, $P_{\mathbb{k}}^R(t, \mathbf{z})$ is coefficientwise bounded above by the right-hand side of (1), as it was shown by Serre. In 1962, Golod [Gol62] characterized Golod rings (hence the name) as those ring where the product and all higher Massey products on the Koszul homology $H_*(K_R)$ are trivial.

In this paper, we only consider *monomial* rings, i.e. quotients of S by a monomial ideal $\mathfrak{a} \subseteq S$. In this setting, the Golod property is related to certain topological features of *moment-angle complexes*. Indeed, if \mathfrak{a} is a squarefree monomial ideal, then it can be interpreted as Stanley-Reisner ideal of a simplicial complex Δ . The moment-angle complex \mathcal{Z}_{Δ} is a certain topological space associated to Δ

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which was introduced by Davis and Januszkiewicz [DJ91]. A prominent feature of this space is that there is an isomorphism of \mathbb{k} -algebras

$$\tilde{H}_*(\mathcal{Z}_\Delta; \mathbb{k}) \cong H_*(K_R),$$

cf. [BP02, Theorem 7.7]. A lot of research was devoted to study the relation between the structure of $H_*(K_R)$ and the topology of \mathcal{Z}_Δ , see for example [DS07; IK14; IK15; GPTW12; BP12]. One of the cornerstones of the study of Golod rings in this context is the following result by Berglund and Jöllenbeck:

Theorem 1.1 (Theorem 5.1, [BJ07]). *Let $\mathfrak{a} \subset S$ be a monomial ideal and let $R := S/\mathfrak{a}$. Then the following are equivalent:*

- (1) *R is Golod.*
- (2) *The product on the Koszul homology of R is trivial.*

Note that the implication $1) \Rightarrow 2)$ holds in general by Golod’s characterization, so the actual content of the theorem is that if the product on $H_*(K_R)$ is trivial, then the higher Massey products are trivial as well. However, we found the following counterexample to this result:

Example 1.2. Let $S = \mathbb{k}[x_1, x_2, y_1, y_2, z, w]$ and let

$$\mathfrak{a} := (x_1x_2^2, z^2w, y_1y_2^2, x_2^2zw, y_2^2z^2, x_1x_2y_1y_2, x_2^2y_2^2z, x_1y_1z) \subset S.$$

For $R := S/\mathfrak{a}$, all products on $H_*(K_R)$ vanish, but R is not Golod. In fact, there is a non-trivial ternary Massey product on $H_*(K_R)$. In terms of moment-angle complexes, this example (after polarization) gives rise to a moment-angle complex with trivial cup-products which is not formal.

We discuss this example in depth in Section 3 below. We would like to point out that the article [BJ07] is in itself correct, but it is building on an incorrect result of [Jö106].

As far as we know, no example of any non-Golod ring with a trivial product on $H_*(K_R)$ was hitherto known, so in fact one could have (quite optimistically) hoped that Theorem 1.1 holds even without the “monomial” assumption.

In view of Example 1.2, it is natural to ask whether one can bound the Massey products one needs to consider. For this we use the following notation: A graded ring R satisfies (B_r) , if all k -ary Massey products are defined and contain only zero for all $k \leq r$. In particular, (B_2) means that all products on the Koszul homology vanish. So a ring R is Golod if and only if it satisfies (B_r) for all $r \in \mathbb{N}$, and Theorem 1.1 stated that for monomial ideals, (B_2) is sufficient for being Golod. Instead, we have the following weaker result:

Theorem 4.1. *Let $\mathfrak{a} \subset S$ be a monomial ideal, let $R := S/\mathfrak{a}$ and let $r := \max(2, \text{reg } R - 2)$. If R satisfies (B_r) , then the ring is Golod.*

As an immediate consequence, we note that Theorem 1.1 holds for simplicial complexes of low dimension:

Corollary 1.3. *Let Δ be a simplicial complex of dimension at most 3. If the product on the Koszul homology of $\mathbb{k}[\Delta]$ is trivial, then $\mathbb{k}[\Delta]$ is Golod.*

Recently, Iriye and Kishimoto [IK15] showed that the Stanley-Reisner ring $\mathbb{k}[\Delta]$ of a triangulated \mathbb{k} -orientable surface Δ is Golod if and only if Δ is 2-neighborly,

i.e. if any two vertices are connected by an edge. The methods we use to prove Theorem 4.1 also give an extension of one implication of that result:

Theorem 4.4. *Let Δ be a triangulation of a \mathbb{k} -orientable manifold. If $\mathbb{k}[\Delta]$ is Golod, then Δ is 2-neighborly.*

This note is structured as follows. In Section 2 some definitions and basic facts concerning Golod rings are recalled. After that, we discuss Example 1.2 in Section 3. In the following Section 4 we prove Theorem 4.1 and Theorem 4.4. Further, in Section 5 we sketch the considerations that lead to us to find Example 1.2. In the last Section 6 two open questions and some remarks are added.

2. PRELIMINARIES

In this section, we recall some facts about Golod rings. We refer the reader to [Avr86] and [Avr98] for a comprehensive treatment of general Golod rings. By a *graded ring*, we always mean a standard graded \mathbb{k} -algebra, i.e. a ring of the form $R = \mathbb{k}[x_1, \dots, x_n]/\mathfrak{a}$ for some homogeneous ideal \mathfrak{a} .

Let us recall the definition of the Massey products of a differential graded algebra (DGA) A . The first Massey product is identically zero and thus usually ignored. The second Massey product of two elements $a_1, a_2 \in H_*(A)$ is just the usual product which is inherited from A . Let us denote it by $\mu_2(a_1, a_2)$. For $n \geq 3$, the n -ary Massey product is a partially defined set-valued function, which assigns to n elements $a_1, \dots, a_n \in H_*(A)$ a set $\mu_n(a_1, \dots, a_n) \subset H_*(A)$. It is defined if there exist elements $a_{ij} \in A$ for $1 \leq i \leq j \leq n$, such that $da_{ii} = 0$, $[a_{ii}] = a_i$ and

$$da_{ij} = \sum_{v=i}^j \bar{a}_{iv} a_{vj},$$

where $\bar{a} = (-1)^{|a|+1}$. Then $\sum_{v=1}^n \bar{a}_{iv} a_{vj}$ is called a Massey product of a_1, \dots, a_n and $\mu_n(a_1, \dots, a_n)$ is the set of all these elements. Further, we say that A satisfies (B_r) , if all k -ary Massey products are defined and contain only zero for all $k \leq r$. We will use the following well-known result.

Lemma 2.1. *Let A be a DGA satisfying (B_{r-1}) . Then $\mu_r(a_1, \dots, a_r)$ is defined and contains only one element for all $a_1, \dots, a_r \in H_*(A)$.*

This is a special case of [May69, Proposition 2.3], see also [Kra66, Lemma 20]. In the second reference, the result is claimed only for classes of odd degree, but the proof also holds in general.

For a graded ring R , we say it satisfies (B_r) if the DGA K_R satisfies (B_r) , where K_R denotes the Koszul complex of R . In particular, (B_2) means that all products on the Koszul homology vanish. By definition, a graded ring R is called *Golod* if the equality Eq. (1) of power series holds. Golod [Gol62] showed that this is equivalent to the condition that all Massey products on the homology $H_*(K_R)$ are trivial, so R is Golod if and only if it satisfies (B_r) for all $r \in \mathbb{N}$.

The Koszul complex K_R of a graded ring carries a natural “homological” \mathbb{N} -grading, in addition to the “internal” grading inherited from R . For a homogeneous element $a \in K_R$, we denote by $|a|$ its homological degree and by $\deg a$ its internal degree.

Let Δ be a simplicial complex with vertex set V . We denote by $\mathbb{k}[\Delta] = S/I_\Delta$ its Stanley-Reisner ring (over some field \mathbb{k}), cf. Chapter 5 of [BH98]. Here, $S = \mathbb{k}[X_v : v \in V(\Delta)]$ is a polynomial ring and

$$I_\Delta = \left(\prod_{v \in M} X_v : M \subseteq V(\Delta), M \notin \Delta \right)$$

is the Stanley-Reisner ideal of Δ . $\mathbb{k}[\Delta]$ carries a natural $\mathbb{N}^{\#V}$ -grading, but in this article we will often consider the (usual) \mathbb{N} -grading inherited from S . For a subset $U \subseteq V$, we denote by

$$\Delta|_U := \{F \in \Delta : F \subseteq U\}$$

the restriction of Δ to U . The following lemma follows easily from considering the multigraded structure of $K_{\mathbb{k}[\Delta]}$

Lemma 2.2. *Let Δ be a simplicial complex such that $\mathbb{k}[\Delta]$ satisfies (B_r) . Then the same holds for every restriction of Δ .*

Finally, we give two simple criteria for the vanishing of products.

Lemma 2.3. *Let $\mathfrak{a} \subseteq S$ be a monomial ideal.*

- (1) *Let $a, b \in H_*(K_R)$ be two Koszul cycles which are homogeneous with respect to the multigrading. If the multidegrees of a and b are not orthogonal (i.e. they have a non-zero component in common), then $a \cdot b = 0$.*
- (2) *Let $m_1, m_2 \in \mathfrak{a}$ be two minimal generators of \mathfrak{a} and let $g_1, g_2 \in K_S$ be two elements in the Koszul complex, such that $\partial g_i = m_i$. If there exists a generator $m_3 \neq m_1, m_2$ of \mathfrak{a} which divides the least common multiple of m_1 and m_2 , then the product $g_1 \cdot g_2$ is zero in $H_*(K_R)$.*

Proof. Both claims are invariant under polarization, so we assume that \mathfrak{a} is square-free. In this case the first claim is clear, because $a \cdot b$ would have a non-squarefree multidegree.

So consider the second claim. If m_1 and m_2 are not coprime then $g_1 \cdot g_2$ is zero by the first claim, hence we may assume that m_1 and m_2 are coprime.

There exist variables $x_a, x_b \in S$ such that x_a divides m_1 , x_b divides m_2 and both do not divide m_3 . As the product on $H_*(K_R)$ does not depend on the choice of representing cycles, we may choose $g_1 = \frac{m_1}{x_a} e_a$ and $g_2 = \frac{m_2}{x_b} e_b$. Here e_a and e_b are the generators of the Koszul complex corresponding to the variables x_a and x_b , respectively. So we have that $g_1 \cdot g_2 = \frac{m_1 m_2}{x_a x_b} e_a \wedge e_b$.

By construction, m_3 divides $\frac{m_1 m_2}{x_a x_b}$ and thus $g_1 \cdot g_2$ is already zero in K_R . \square

3. DISCUSSION OF THE EXAMPLE

In this section we discuss why Example 1.2 has the claimed properties, namely that

- (1) the product on $H_*(K_R) = \text{Tor}_*^S(S/\mathfrak{a}, \mathbb{k})$ is trivial, and that
- (2) S/\mathfrak{a} is not Golod.

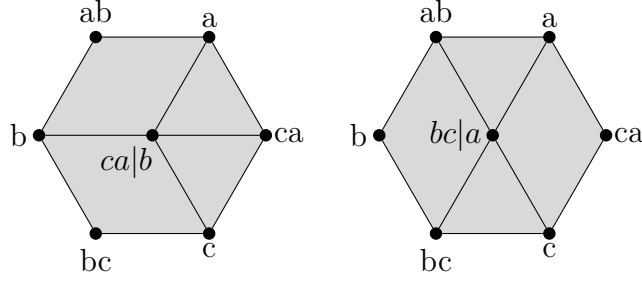


FIGURE 1. The cellular resolution of S/\mathfrak{a} . This is a 3-ball with one 3-cell and the figure shows the top and the bottom of this ball.

For the remainder of this section, let $S = \mathbb{k}[x_1, x_2, y_1, y_2, z, w]$ and let $\mathfrak{a} \subset S$ the ideal of Example 1.2, i.e. the ideal with the following generators:

$$\begin{array}{lll} m_a := x_1 x_2^2 & m_{ab} := x_2^2 z w & m_{bc|a} := x_2^2 y_2^2 z \\ m_b := z^2 w & m_{bc} := y_2^2 z^2 & m_{ca|b} := x_1 y_1 z \\ m_c := y_1 y_2^2 & m_{ca} := x_1 x_2 y_1 y_2 & \end{array}$$

The naming convention of the generators is explained in Section 5 below. For $\epsilon \in \{a, b, c, ab, bc, ca, bc|a, ca|b\}$ let g_ϵ denote the element of the Koszul complex with $\partial g_\epsilon = m_\epsilon$. The minimal free resolution of S/\mathfrak{a} is cellular (in the sense of [BS98]), see Fig. 1. Using Macaulay2 or Fig. 1, one can compute the Betti diagram of S/\mathfrak{a} , see Table 1. From this one can read off that for degree reasons, the only products which can possibly be non-zero are the following:

- (1) $g_i \cdot g_j$ for $i, j \in \{a, b, c\}, i \neq j$,
- (2) $g_i \cdot g_{ca|b}$ for $i \in \{a, b, c\}$, and
- (3) $g_i \cdot h$ for $i \in \{a, b, c, ca|b\}$ and h is a generator with $|h| = 3$ and $\deg h = 6$, i.e. the “2” in Table 1.

It holds that $m_{ij} \mid \text{lcm}(m_i, m_j)$ for $(i, j) \in \{(a, b), (b, c), (c, a)\}$, hence these products are zero by Lemma 2.3. Moreover, $m_{ca|b}$ is not coprime with m_a, m_b or m_c , so the second class of products are zero for degree reasons. Finally, it is not difficult to see that the generators h with $|h| = 3$ and $\deg h = 6$ correspond to the triangles $\{a, ca, ca|b\}$ and $\{c, ca, ca|b\}$ in Fig. 1. So their multidegrees are not orthogonal to the multidegree of any g_i for $i \in \{a, b, c, ca|b\}$, and hence this products are zero as well. Thus, all products on $\text{Tor}_*^S(S/\mathfrak{a}, \mathbb{k})$ vanish. Alternatively, one can use the Macaulay2 command `isHomologyAlgebraTrivial` from the package `DGAlgebras` by Frank Moore (which is distributed along with Macaulay2) to verify this.

To see that the ring is not Golod, we compare its Poincaré-Betti series with the expected series, i.e. the right-hand side of Eq. (1). The Poincaré-Betti series of $R := S/\mathfrak{a}$ can be computed using the formula given in [Ber06, Theorem 1]. It is

$$P_{\mathbb{k}}^R(t, z) = \frac{(1 + zt)^6}{1 - t((4z^3 + 3z^4 + z^5)t + (10z^5 + 4z^6)t^2 + (2z^6 + 6z^7)t^3 - z^9t^5)}.$$

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|----|---|---|
| 0 | 1 | . | . | . | . |
| 1 | . | . | . | . | . |
| 2 | . | 4 | . | . | . |
| 3 | . | 3 | 10 | 2 | . |
| 4 | . | 1 | 4 | 6 | . |
| 5 | . | . | . | . | 1 |

TABLE 1. The Betti diagram of S/\mathfrak{a} . The entry in position (i, j) is $\beta_{i,i+j}^S(S/\mathfrak{a})$ and the dots stand for zeros.

On the other hand, the right-hand side of Eq. (1) can be read off from Table 1. It equals

$$\frac{(1 + zt)^6}{1 - t((4z^3 + 3z^4 + z^5)t + (10z^5 + 4z^6)t^2 + (2z^6 + 6z^7)t^3 + z^9t^4)},$$

so the series are different and thus R is not Golod. Alternatively, one can use `Macaulay2` to compute the first terms of the Poincaré-Betti series by computing the beginning of a resolution of \mathbb{k} over R . In fact, with a little more work one can show that the ternary Massey product $\mu_3(g_a, g_b, g_c)$ is non-zero. It corresponds to the 1 in the lower right corner of the Betti diagram.

Remark 3.1. The polarization of the ideal in Example 1.2 is the Stanley-Reisner ideal of some simplicial complex Δ of dimension 5. By taking its 4-skeleton, one obtains an example of a 4-dimensional simplicial complex Γ , such that $\mathbb{k}[\Gamma]$ is not Golod but has a the product on its Koszul homology. Indeed, the product stays trivial under taking the skeleton by [Kat15, Corollary 5.1], and the non-vanishing Massey product is a 4-cycle, so it cannot become a boundary when we remove simplices of higher dimension from Δ . This shows that Corollary 1.3 is best possible.

In the end of this section, we briefly discuss what we consider to be the reason for the failure of Theorem 1.1. First, this result was stated in [Jöl06, Theorem 7.1] under the additional assumption that $R = S/\mathfrak{a}$ satisfies a certain property **(P)**. In that article, it was conjectured that every monomial ring has this property, and that conjecture was then confirmed in [BJ07], leading to the unconditional statement of Theorem 1.1 in [BJ07, Theorem 5.1].

We have no reason to doubt the validity of the main result of [BJ07]. Instead, the problem lies in [Jöl06, Theorem 7.1]. The proof of this result goes by applying discrete Morse theory to the Taylor resolution of S/\mathfrak{a} . Here, a special type of Morse matching on the Taylor complex is used, a so-called *standard matching*. We refer the reader to [Jöl06] for the precise definition. A standard matching is compatible with the multiplicative structure of the Taylor complex in a certain way. This compatibility is crucial for the study of the multiplicative structure on $\mathrm{Tor}_*^S(S/\mathfrak{a}, \mathbb{k})$.

It is stated in [Jöl06] that such a standard matching always exists. But this is not true. For example, it is not difficult to see from the definition that the ideal

$$\mathfrak{a} = (x_1^2, x_1x_2, x_2x_3, x_3x_4, x_4^2) \subset \mathbb{k}[x_1, \dots, x_4] = S$$

does not allow a standard matching. In fact, this example is taken from [Avr98, Theorem 2.3.1] where it is given as an example of an ideal whose minimal free resolution does not allow a DGA structure. A standard matching does generally not induce a DGA structure, but it is related. So in this example at least, the non-existence of a DGA structure seems to prevent a standard matching as well. As this example does not satisfy (B_2) , it does not directly yield a counterexample to Theorem 1.1.

4. A BOUND IN TERMS OF THE REGULARITY

Given the failure of Theorem 1.1, we show the following weaker version of that result:

Theorem 4.1. *Let $\mathfrak{a} \subset S$ be a monomial ideal, let $R := S/\mathfrak{a}$ and let $r := \max(2, \text{reg } R - 2)$. If R satisfies (B_r) , then the ring is Golod.*

We prove this at the end of this section. The following lemma is the key step in our proof of Theorem 4.1.

Lemma 4.2. *Let Δ be a simplicial complex. Assume that there exists a nonzero cohomology class $\alpha \in H^d(\Delta; \mathbb{k})$ for some $d \geq 1$, such that the restriction of α to any induced subcomplex of Δ is zero.*

If Δ is not 2-neighborly, then $\mathbb{k}[\Delta]$ does not satisfy (B_2) , i.e. there exists a non-vanishing product on the Koszul homology.

Here, “2-neighborly” means that every two vertices are connected by an edge.

Proof. Let V be the vertex set of Δ . For $i \in \mathbb{N}$ and two non-empty disjoint subsets $I, J \subset V$, we write $\varphi_i^{I,J} : \tilde{H}_i(\Delta|_{I \cup J}; \mathbb{k}) \rightarrow \tilde{H}_i(\Delta|_I * \Delta|_J; \mathbb{k})$ for the map induced by the inclusion $\Delta|_{I \cup J} \hookrightarrow \Delta|_I * \Delta|_J$. Recall that these maps vanish if and only if all products on the Koszul homology vanish [IK14]. For a chain $\tau = \sum_{\sigma \in \Delta} c_\sigma \sigma$ on Δ we define

$$V(\tau) := \{v \in V : \exists \sigma \in \Delta : v \in \sigma, c_\sigma \neq 0\}.$$

Our hypothesis implies that

$$(2) \quad \langle \omega, \alpha \rangle = 0 \text{ for all } \omega \in H_d(\Delta) \text{ with } V(\omega) \neq V.$$

If Δ is not 2-neighborly, then there exist two vertices v, w which are not connected by an edge. Set $I := \{v, w\}$ and $J := V \setminus I$. Note that $d \geq 1$ implies that Δ has more than two vertices and hence $J \neq \emptyset$. As $\alpha \neq 0$, there exists an $\omega \in H_d(\Delta)$ such that $\langle \omega, \alpha \rangle \neq 0$. We claim that $\varphi_d^{I,J}(\omega) \neq 0$, and hence there exists a non-vanishing product on the Koszul homology.

Before we prove the claim, we define some auxiliary maps. For a chain $\tau = \sum_{\sigma} c_\sigma \sigma$ on Δ we set

$$\begin{aligned} \tau_v &:= \sum \{c_\sigma \sigma : v \in \sigma\} \quad \text{and} \\ \tau^v &:= \sum \{c_\sigma (\sigma \setminus \{v\}) : v \in \sigma\}. \end{aligned}$$

Choose a linear order on V such that v is the smallest vertex and orient Δ accordingly. Under this convention it holds that $d\tau_v = (d\tau)_v + \tau^v$.

Now we turn to the proof of our claim. Assume the contrary, i.e. that $\varphi_d^{I,J}(\omega) = 0$. Then there exists a $(d+1)$ -chain τ in $\Delta|_I * \Delta|_J$ such that $d\tau = \omega$. Set

$\omega' := d\tau_v = \omega_v + \tau^v$. Note that ω' is in fact a chain on Δ , because both ω_v and τ^v are chains on Δ . (For the latter, recall that $\tau \in \Delta|_I * \Delta|_J$ and $\Delta|_I$ are just the two disconnected vertices v and w .) Further, $v \notin V(\omega - \omega')$, hence by Eq. (2) it follows that $\langle \omega - \omega', \alpha \rangle = 0$ and thus

$$\langle \omega', \alpha \rangle = \langle \omega, \alpha \rangle \neq 0.$$

As v and w are not connected in $\Delta|_I * \Delta|_J$, it holds that $w \notin V(\tau_v)$. Hence $w \notin V(\omega')$ as well, contradicting Eq. (2). \square

From the preceding lemma, we obtain the following result, which we consider to be of independent interest:

Proposition 4.3. *Let $\mathfrak{a} \subset S$ be a monomial ideal, let $R := S/\mathfrak{a}$ and assume that R satisfies (B_{r-1}) . Let $a_1, \dots, a_r \in H_*(K_R)$ be elements of the Koszul homology. If $\deg a_i = |a_1| + 1$ for some i (i.e. a_i lies in the 2-linear strand of K_R), then $\mu_r(a_1, \dots, a_r)$ contains only zero.*

Proof. The claim is invariant under polarization, so we may assume that \mathfrak{a} is squarefree and thus the Stanley-Reisner ideal of some simplicial complex Δ with vertex set V . Assume to the contrary that $\mu_r(a_1, \dots, a_r)$ contains a non-zero element α . There exists a $U \subset V$ and $d \in \mathbb{N}$ such that α corresponds to a cohomology class $\alpha \in H^d(\Delta|_U)$.

For any subset $U' \subseteq U$, the inclusion $\Delta|_{U'} \hookrightarrow \Delta|_U$ induces a map $r_{U,U'} : H_*(K_{\mathbb{k}[\Delta|_U]}) \rightarrow H_*(K_{\mathbb{k}[\Delta|_{U'}]})$ which is compatible with Massey products. So by replacing U by a suitable subset, we may assume that $r_{U,U'}(\alpha) = 0$ for every proper subset of U .

Note that $\Delta|_U$ satisfies (B_{r-1}) because Δ does (cf. Lemma 2.2), and α is also a Massey product of $\Delta|_U$. So we may replace Δ by $\Delta|_U$.

Assume that $\deg a_i = |a_1| + 1$ for some i . The element a_i is nonzero, because otherwise zero would be contained in $\mu_r(a_1, \dots, a_r)$. But by Lemma 2.1, this set contains only one element α and we assumed that to be non-zero. The element a_i corresponds to a 0-class in some restriction of Δ , and as it is non-zero, this restriction is disconnected. But now Δ satisfies the hypothesis of Lemma 4.2 and it is not 2-neighborly, thus $\mathbb{k}[\Delta]$ does not satisfy (B_2) , a contradiction. \square

To deduce Theorem 4.1 from Proposition 4.3, one only needs degree considerations.

Proof of Theorem 4.1. Assume that R satisfies (B_{r-1}) but not (B_r) for some $r \geq 3$. Then there exist homogeneous elements $a_1, \dots, a_r \in H_*(K_R)$ such that $\alpha \in \mu_r(a_1, \dots, a_r)$ is nonzero. Now Proposition 4.3 implies that $\deg a_i \geq |a_i| + 2$ for all i . Hence

$$\deg \alpha = \sum_i \deg a_i \geq \sum_i |a_i| + 2r = |m| + r + 2$$

Thus, $\alpha \neq 0$ implies that $r \leq \operatorname{reg}_S R - 2$. \square

We close this section with another consequence of Lemma 4.2. It extends the result on Golod surfaces by Iriye and Kishimoto [IK15, Theorem 1.3].

Theorem 4.4. *Let Δ be a triangulation of a \mathbb{k} -orientable manifold. If $\mathbb{k}[\Delta]$ is Golod, then Δ is 2-neighborly.*

Proof. The fundamental class of Δ satisfies the hypothesis of Lemma 4.2. \square

The converse of this result holds if Δ is two-dimensional, cf. [IK15]. In higher dimensions, the converse does not hold. Indeed, the boundary complex of any simplicial 2-neighborly polytope is Gorenstein* and thus not Golod.

Moreover, the assumption that Δ is \mathbb{k} -orientable cannot be removed. For example let Δ be the complex obtained from the usual 6-vertex triangulation of the real projective plane by applying a stellar subdivision to one of the facets. Then Δ is not 2-neighborly but $\mathbb{k}[\Delta]$ is Golod if (and only if) $\text{char } \mathbb{k} \neq 2$. See [Kat15, Example 4.2] for a detailed discussion of this example.

5. HOW THE EXAMPLE WAS FOUND

In this section we sketch the considerations that lead to Example 1.2. Our Ansatz is to construct a squarefree monomial ideal \mathfrak{a} , such that all products in the Koszul homology are trivial and that there exist three Koszul cycles of homological degree 1 whose ternary Massey product is nonzero. For simplicity we work in characteristic 2, so we do not have to keep track of signs. The Example 1.2 we found is nevertheless independent of the characteristic.

We compute $\text{Tor}_*^S(S/\mathfrak{a}, \mathbb{k})$ via the *Taylor resolution* T_\bullet of S/\mathfrak{a} , cf. [MS05, p. 67]. Recall that the Taylor complex is the complex of free S -modules with basis $\{e_I : I \subseteq G(\mathfrak{a})\}$, where $G(\mathfrak{a})$ denotes the set of minimal generators of \mathfrak{a} . The basis elements are graded by $|e_I| := \#I$ and $\deg e_I := \deg m_I$, where $m_I := \text{lcm}(m : m \in I)$ for $I \subseteq G(\mathfrak{a})$. Further, the differential is given by

$$\partial e_I = \sum_{m \in I} \frac{m_I}{m_{I \setminus \{m\}}} e_{I \setminus \{m\}}$$

and T_\bullet carries a DGA structure with the multiplication given by

$$e_I \cdot e_J = \frac{m_I m_J}{m_{I \cup J}} e_{I \cup J}.$$

We identify squarefree monomials with finite sets. If we want to explicitly turn a finite set a into a monomial we write $m_a := \prod_{i \in a} x_i$. So we have three finite sets a, b, c , corresponding to generators of our ideal, such that $\mu_3(g_a, g_b, g_c)$ should not be zero. This implies already that a, b, c are pairwise disjoint. Further, we want that the product between any two elements is zero. So by Lemma 2.3 we need additional generators ab, bc, ca of our ideal, such that $ab \subset a \cup b$, $bc \subset b \cup c$ and $ca \subset c \cup a$. Now a, b and c being disjoint implies that ab is disjoint from c , and similarly $a \cap bc = b \cap ca = \emptyset$. Hence we need three further generators $ab|c, bc|a, ca|b$ with $ab|c \subset ab \cup c$, $bc|a \subset bc \cup a$ and $ca|b \subset ca \cup b$.

Altogether, we have nine generators $a, b, c, ab, bc, ca, ab|c, bc|a, ca|b$ satisfying

- (1) a, b, c are pairwise disjoint.
- (2) $ij \subset i \cup j$ and $ij \cap i, ij \cap j \neq \emptyset$ for $(i, j) \in \{(a, b), (b, c), (c, a)\}$.
- (3) $ij|k \subset ij \cup k$ and $ij|k \cap ij, ij|k \cap k \neq \emptyset$ for $(i, j, k) \in \{(a, b, c), (b, c, a), (c, a, b)\}$.

Note that the three last generators might be equal to each other or to some of the ij -generators. In fact, in Example 1.2 we have that $ab|c = bc|a$.

Next, we compute the Massey product $\mu_3(g_a, g_b, g_c)$. For this we use the DGA structure on $T_\bullet \otimes \mathbb{k}$. It holds that

$$g_a \cdot g_b = g_{a,b} = \partial g_{a,ab,b}.$$

For the last equation, we use that $a \setminus ab \neq \emptyset$ and $b \setminus ab \neq \emptyset$. So the corresponding terms of $\partial g_{a,ab,b}$ have a non-constant monomial factor, which is zero in $T_\bullet \otimes \mathbb{k}$. By a similar computation for $g_b \cdot g_c$, we obtain that

$$\mu_3(g_a, g_b, g_c) = g_{a,ab,b}g_c + g_ag_{b,bc,c} = g_{a,ab,b,c} + g_{a,b,bc,c}.$$

We need to make sure that this is non-zero. It holds that

$$\partial g_{a,ab,b,bc,c} = m_{a \setminus ab}g_{ab,b,bc,c} + g_{a,b,bc,c} + m_{b \setminus (ab \cup bc)}g_{a,ab,bc,c} + g_{a,ab,b,c} + m_{c \setminus bc}g_{a,ab,b,bc}.$$

The first and the last term are zero, because $a \setminus ab \neq \emptyset$ and $c \setminus bc \neq \emptyset$. Hence

$$\mu_3(g_a, g_b, g_c) = g_{a,ab,b,c} + g_{a,b,bc,c} = m_{b \setminus (ab \cup bc)}g_{a,ab,bc,c}.$$

For this to be non-zero, it is necessary that $b \setminus (ab \cup bc) = \emptyset$, or equivalently

$$(3) \quad b \subset ab \cup bc$$

Next, we compute

$$\begin{aligned} \partial g_{a,ab,ca,bc,c} &= m_{a \setminus (ab \cup ca)}g_{ab,ca,bc,c} + m_{ab \setminus (a \cup bc)}g_{a,ca,bc,c} + g_{a,ab,bc,c} \\ &\quad + m_{bc \setminus (ab \cup c)}g_{a,ab,ca,c} + m_{c \setminus (bc \cup ca)}g_{a,ab,ca,bc} \end{aligned}$$

Equation (3) implies that $ab \setminus (a \cup bc) \neq \emptyset$ and $bc \setminus (ab \cup c) \neq \emptyset$, so the corresponding terms are zero. So for the Massey product to be nonzero, it is necessary that $a \subset ab \cup ca$ or $c \subset bc \cup ca$. By symmetry, we only consider the first case, so assume that

$$(4) \quad a \subset ab \cup ca$$

We claim that Eq. (3) and Eq. (4) imply that $ab|c$ is different from a, b, c, ab, bc and ca . Indeed, it is clearly different from a, b, c and ab . If $ab|c = ca$, then Eq. (4) implies that $a \subseteq ab \cup ca = ab \cup ab|c = ab$, a contradiction to the assumption that a is a minimal generator of \mathfrak{a} . Hence $ab|c \neq ca$ and similarly, Eq. (3) implies that $ab|c \neq bc$. In the next step, we consider

$$(5) \quad \begin{aligned} \partial g_{ab,ca,bc,c,ab|c} &= m_{ab \setminus (ca \cup bc \cup ab|c)}g_{ca,bc,c,ab|c} + m_{ca \setminus (ab \cup bc \cup c \cup ab|c)}g_{ab,bc,c,ab|c} \\ &\quad + m_{bc \setminus (ab \cup ca \cup c \cup ab|c)}g_{ab,ca,c,ab|c} + m_{c \setminus (ab \cup ca \cup bc \cup ab|c)}g_{ab,ca,bc,ab|c} + g_{ab,ca,bc,c} \end{aligned}$$

We claim that $ca \setminus (ab \cup bc \cup c \cup ab|c) \neq \emptyset$. Otherwise, we would have

$$ca \subseteq ab \cup bc \cup c \cup ab|c$$

and hence $ca \subseteq c \cup ab$. But this contradicts Eq. (4), because a and c are disjoint. By exchanging a and b in this argument, one also sees that $bc \setminus (ab \cup bc \cup c \cup ab|c) \neq \emptyset$. Again, for the Massey product to be non-zero, at least one of the monomial coefficients in Eq. (5) has to be constant. So it follows that

$$c \subseteq ca \cup bc \cup ab|c \quad \text{or} \quad ab \subseteq ca \cup bc \cup ab|c$$

From here on, the combinatorial conditions we derive become more and more complicated, with more and more cases. So we used a computer to automate the reasoning we just demonstrated, and considered all possible cases in each step. To check whether the conditions are sufficient to ensure that the Massey product is nonzero, we considered sets a, b, \dots “as generic as possible” with respect to the given constraints and computed the Massey product in $T_\bullet \otimes \mathbb{k}$.

A rather short computer search yielded several examples of choices of the generators such that the Massey product is non-zero. Note that the examples found this way are guaranteed to have trivial products of elements of homological degree

1 and to have $\mu_3(g_a, g_b, g_c) \neq 0$. However, one has to check separately that there are no other non-zero products (in higher homological degrees). Finally, Example 1.2 was obtained from such a computer-generated example by the deletion of some variables and by de-polarizing.

To close the section, we note the following observation:

Proposition 5.1. *Let $\mathfrak{a} \subseteq S$ be a monomial ideal with at most seven generators. If the product on the Koszul homology of S/\mathfrak{a} is trivial, then S/\mathfrak{a} is Golod.*

This follows from considerations similar to those above. It is not difficult to see that a counterexample to Theorem 1.1 with the minimal number of generators needs to be of the form of our Ansatz. Hence our considerations already show that one needs at least seven different generators, namely a, b, c, ab, bc, ca and $ab|c$. So see that in fact eight generators are needed, one assumes that one only needs seven. Then $ca|b$ and $bc|a$ have to be equal to two of the already mentioned generators. An extensive case distinction then leads to a contradiction. As this proof is very technical and not very illuminating, we omit the details.

6. QUESTIONS AND REMARKS

In this last section, we collect some open questions and some remarks.

6.1. A general bound for the Massey products. We wonder whether Theorem 4.1 holds more generally:

Question 6.1. Is the assumption “monomial” in Theorem 4.1 really necessary? More precisely, let $\mathfrak{a} \subset S$ be a homogeneous ideal, let $R := S/\mathfrak{a}$ and let $r := \max(2, \text{reg } S/\mathfrak{a} - 2)$. If R satisfies (B_r) , does it follow that R is Golod?

Note that to answer this question, it would be enough to prove Proposition 4.3 for general graded rings. For completeness, we also note the following criteria for the Golod property, which appear similar to Theorem 4.1 but are actually rather trivial:

Proposition 6.2. *Let $\mathfrak{a} \subset S = \mathbb{k}[x_1, \dots, x_n]$ be a homogeneous ideal.*

- (1) *If S/\mathfrak{a} satisfies $(B_{\lfloor p/2 \rfloor + 1})$ for $p = \text{pdim } S/\mathfrak{a}$, then it is Golod.*
- (2) *If \mathfrak{a} is a squarefree monomial ideal which contains no variable and S/\mathfrak{a} satisfies $(B_{\lfloor n/2 \rfloor})$, then S/\mathfrak{a} is Golod.*

Proof. The first claim follows easily by considering the homological degree of a Massey product.

For the second claim we consider the multigrading. Under the given assumption, every nonzero Koszul cycle has at least two nonzero components in its multidegree. So any Massey product of more than $\lfloor n/2 \rfloor$ factors is zero for degree reasons. \square

6.2. Few variables. Example 1.2 has six variables and we conjecture that this is optimal:

Conjecture 6.3. Let $\mathfrak{a} \subseteq S$ be a monomial ideal and assume that $\dim S \leq 5$. If the product on the Koszul homology of S/\mathfrak{a} is trivial, then S/\mathfrak{a} is Golod.

It is not difficult to see that any counterexample \mathfrak{a} to Theorem 1.1 needs to have at least five variables. Indeed, any such counterexample has at least three generators which are pairwise coprime. Further, if we apply the considerations of the last section to the polarization of \mathfrak{a} , then Eq. (3) and Eq. (4) imply that the generators corresponding to a and b are not pure powers. Thus \mathfrak{a} has at least $2 + 2 + 1 = 5$ variables. For a proof that indeed six variables are needed, it remains to show that that last generator is not a pure power as well. The squarefree situation is easier:

Proposition 6.4. *Any simplicial complex whose Stanley-Reisner ideal contradicts Theorem 1.1 has at least nine vertices.*

The polarization of Example 1.2 has nine variables, so this bound is sharp.

Proof. Again, any counterexample to Theorem 1.1 has at least three generators which are pairwise coprime. Further, by Proposition 4.3 we may also assume that these generators have degree at least three. But three squarefree pairwise coprime monomials of degree at least three can only exist if the ambient ring has at least nine variables. \square

6.3. The Golod property of flag simplicial complexes. An important application of Theorem 1.1 is the characterization of the Golod property among *flag* simplicial complexes, [BJ07, Theorem 6.4]. The proof of this result given in [BJ07] depends on Theorem 1.1, but there is an independent proof by Grbic, Panov, Theriault and Wu [GPTW12, Theorem 4.6]. Hence this characterization is not affected by the failure of Theorem 1.1.

6.4. Generic monomial ideals. Finally, note that the statement of Theorem 1.1 holds for *generic* monomial ideals [BPS98; MSY00]. It is well-known that the minimal free resolution of a generic monomial ideal carries a DGA structure [BPS98, Corollary 3.6]. So under this hypothesis, Theorem 1.1 follows from Proposition 5.2.4(4) of [Avr98].

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